B.Sc III Paper - V

Que :— State and Proove Existence theorem and uniform limit

Sol^n :—

Existence Theorem :— let $F(x,y)$ be a real valued Function of variables x, y where (x,y) varies over $S \times T \subseteq R^2$ and let $a \in \bar{S}, b \in \bar{T}$

If $\lim_{\substack{x \to a \\ y \to b}} F(x,y) = l$ exists and if any inner limit $\lim_{y \to b} F(x,y)$

$(\lim_{x \to a} F(x,y))$ exists then corresponding repeated limit l_{21} (l_{12}) exists and is equl to l.

proof :- Let $\lim_{y \to b} F(x,y) = l_2(x), x \in S$

Since also $\lim_{\substack{x \to a \\ y \to b}} F(x,y) = l$ (by hypothesis) for any $\epsilon > 0 \; \exists V(a) \; \& \; \exists V(b)$

such that
$$|F(x,y) - l| < \frac{\epsilon}{2} \text{ and } |F(x,y) - l_2(x)| < \frac{\epsilon}{2}$$
For $x \in V(a) \; \& \; y \in V(b)$ — (1)

Now $|l_2(x) - l| \le |l_2(x) - F(x,y)| + |F(x,y) - l|$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{—— (II)}$$
provided $x \in V(a) \; \& \; y \in V(b)$

But since the L.H.S expression is independent of y result (ii) means that for any $\varepsilon > 0 \; \exists \; V(a)$ such that

$$|l_2(x) - l| < \varepsilon, \; x \in V(a) \quad \therefore \; l_{21} = \lim_{x \to a} l_2(x) = l$$

Thus Completes the proof.

Def. Uniform limit:—

Suppose $\lim_{x \to a} f(x,y) = g(y), \; y \in T$

This means that for each fixed $y \in T$ corresponding to any $\varepsilon > 0$, $\exists \; \delta(y)$ such that

$$|f(x,y) - g(y)| < \varepsilon, \text{ provided } |x - a| < \delta(y)$$

Symbolically we express this as follows:—

$$(\varepsilon > 0)\,(y)\,(E\,\delta)\,[\,|f(x,y) - g(y)| < \varepsilon \text{ provided } |x - a| < \delta\,]$$

Here as is obvious δ depends not only upon ε but upon the y as well.

However if in certain cases δ depends upon ε only (but not upon y)

i.e if $(\varepsilon > 0)(\exists \delta)(y)\,[\,|f(x,y) - g(y)| < \delta$ provided $|x - a| < \delta\,]$

then we say that

$$\lim_{x \to a} f(x,y) = g(y)$$

uniformly over T.